Written Exam Economics Summer 2021

Advanced Financial and Macro Econometrics

June 22, 10:00 – 22:00

This exam question consists of 6 pages in total

Answers only in English. A take-home exam paper cannot exceed 10 pages (plus a few additional pages with output and supplementary material). One page is defined as 2400 keystrokes.

The paper must be uploaded as one PDF document. The PDF document must be named with exam number only (e.g. '1234.pdf') and uploaded to Digital Exam.

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1 Co-integration and Pricing

Let X_t and Y_t denote vectors of stock prices. Consider two co-integrated VAR systems of each $p = 3$ variables, $X_t = (x_{1,t}, x_{2,t}, x_{3,t})'$ and $Y_t = (y_{1,t}, y_{2,t}, y_{3,t})'$, as given by

$$\Delta X_t = \alpha\beta' X_{t-1} + \epsilon_t \tag{1.1}$$

$$\Delta Y_t = ab' Y_{t-1} + e_t, \tag{1.2}$$

with ϵ_t i.i.d. N(0,Ω) and e_t i.i.d. N(0,O) where Ω and O are two 3×3 positive definite matrices. You are informed that the following equilibrium relationships exist:

$$x_{1,t} - \tfrac{1}{4}x_{2,t} = u_{1,t} \tag{1.3}$$

$$x_{2,t} + \tfrac{3}{4}x_{3,t} = u_{2,t} \tag{1.4}$$

$$y_{1,t} - y_{3,t} = u_{3,t}, \tag{1.5}$$

where $u_{i,t}$ is a stationary process, $i = 1, 2, 3$. Both systems fulfill the assumptions for the Granger representation theorem,

$$X_t = \beta_\perp \sum_{i=1}^{t} \tilde{\alpha}'_\perp \epsilon_i + \varrho_t + \Lambda \tag{1.6}$$

$$Y_t = b_\perp \sum_{i=1}^{t} \tilde{a}'_\perp e_i + S_t + A, \tag{1.7}$$

where $\beta_\perp$ and $b_\perp$ contain the loadings to the common stochastic trends given by $\sum_{i=1}^{t} \tilde{\alpha}'_\perp \epsilon_i$ and $\sum_{i=1}^{t} \tilde{a}'_\perp e_i$, respectively, with

$$\tilde{\alpha}'_\perp = (\alpha'_\perp \beta_\perp)^{-1} \alpha'_\perp \quad \text{and} \quad \tilde{a}'_\perp = (a'_\perp b_\perp)^{-1} a'_\perp. \tag{1.8}$$

In addition, ϱ_t and S_t are stationary process, and Λ and A are functions of initial values.

[1] For each model, (1.1) and (1.2), state the number of stochastic trends.
Write the loading matrices, $\beta_\perp$ and $b_\perp$, implied by the information given above.

[2] Give an expression for ϱ_t and S_t.

[3] Explain how the two observed time series, $\{X_t\}_{t=0}^T$ and $\{Y_t\}_{t=0}^T$, can be used to form a pairs-trading portfolio.

[4] Consider a combined system of four variables, $Z_t = (x_{1,t}, x_{3,t}, y_{1,t}, y_{3,t})'$.

[4.1] State the co-integrating relationships in the model for Z_t.
For the Granger representation corresponding to the model for Z_t,

$$Z_t = \beta^*_\perp \sum_{i=1}^{t} \alpha^{*\prime}_\perp \eta_i + S^*_t + A^*, \tag{1.9}$$

state the loading matrix to the common stochastic trends, $\beta^*_\perp$.

[4.2] Now, you are informed that there exists co-integration also between the two systems in (1.1) and (1.2), such that

$$x_{1,t} - y_{3,t} = u_{4,t}, \tag{1.10}$$

where $u_{4,t}$ is a stationary process.
State the co-integration relationships in the model for Z_t and the loading matrix for the common stochastic trends, $\beta_{\perp}^{*}$, for this case.

[5] Focus on the marginal system for $Y_t = (y_{1,t}, y_{2,t}, y_{3,t})'$ and let $y_{3,t}$ denote the market portfolio.

[5.1] Derive the short-run conditional pricing beta for the first asset

$$B_t^1 = \frac{\mathrm{cov}_{t-1}(\Delta y_{1,t}, \Delta y_{3,t})}{V_{t-1}(\Delta y_{3,t})}, \tag{1.11}$$

where $\mathrm{cov}_{t-1}(\cdot,\cdot) = \mathrm{cov}(\cdot,\cdot \mid \mathcal{F}_{t-1})$ and $V_{t-1}(\cdot) = V(\cdot \mid \mathcal{F}_{t-1})$, with $\mathcal{F}_{t-1} = \{Y_{t-1}, Y_{t-2}, Y_{t-3}, ...\}$ being the natural filtration.
Explain why, or why not, B_t^1 is time varying.
Explain how B_t^1 can be estimated in the conditional model for $y_{1,t}, y_{2,t} \mid y_{3,t}$.
Derive the conditions on the parameters in (1.2) under which B_t^1 can be consistently estimated in a linear regression of $\Delta y_{1,t}$ on $\Delta y_{3,t}$.

[5.2] Explain how the beta, B_t^1, can be used for asset pricing, and explain the type of risk the agent is compensated for according to the CAPM equilibrium pricing model and the type of risk he is not compensated for.

[5.3] The pairs-trading strategy is often celebrated as *market neutral*. Explain what that means in terms of a pricing beta and compare with the pricing based on B_t^1.

[6] Finally, consider the extended model

$$\Delta Y_t = \mu + ab'Y_{t-1} + e_t \tag{1.12}$$

$$e_t = O_t^{1/2}\eta_t \tag{1.13}$$

$$O_t = D_t \Gamma D_t, \tag{1.14}$$

where $D_t = \mathrm{diag}(\sigma_{1,t}, \sigma_{2,t}, \sigma_{3,t})$, with ARCH(1) diagonals,

$$\sigma_{i,t}^2 = \omega_i + \lambda_i e_{i,t-1}^2, \quad i = 1, 2, 3, \tag{1.15}$$

Γ is a 3×3 constant correlation matrix

$$\Gamma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}, \tag{1.16}$$

η_t is i.i.d. N(0,I_3), and

$$b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{1.17}$$

Explain why, or why not, the pricing beta, B_t^1, is time varying, and how it can be estimated. Show that it can be written as

$$B_t^1 = \rho_{13} \frac{\sigma_{1,t}}{\sigma_{3,t}}. \tag{1.18}$$

[7] The data file `Exam_2021.xls` contains the three variables $y_{1,t}$, $y_{2,t}$, $y_{3,t}$, their first differences, $\Delta y_{1,t}$, $\Delta y_{2,t}$, $\Delta y_{3,t}$, and the lagged spread, $y_{1,t-1} - y_{3,t-1}$, for $t = 1, 2, ..., 400$.
Estimate the model outlined above and report parameter estimates.
Report estimated conditional variances, covariances and correlations.
Report also estimated time-varying betas, B_t^1 and B_t^2.
Comment on the results. Argue, in particular, if this model is a good starting point for calculating time-varying betas.

2 Multivariate GARCH-X

Consider the bivariate CCC model for $Y_t = (y_{1,t}, y_{2,t})'$ given by:

$$Y_t = \Omega_t^{1/2} z_t, \quad t = 1, 2, ..., T. \tag{2.1}$$

Here z_t is an i.i.d. N$(0, I_2)$ innovations sequence, and

$$\Omega_t = D_t \Gamma D_t, \tag{2.2}$$

where

$$\Gamma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \text{and} \quad D_t = \text{diag}\left(\sigma_{i,t}\right)_{i=1,2} = \begin{pmatrix} \sigma_{1,t} & 0 \\ 0 & \sigma_{2,t} \end{pmatrix}. \tag{2.3}$$

Moreover, the conditional variances are given by,

$$\sigma_{i,t}^2 = \omega_i + \alpha_i y_{i,t-1}^2 + \gamma x_{t-1}^2 \tag{2.4}$$

where x_t is an exogenous covariate. *Note that γ is the same parameter for both conditional variances.*

The parameter vector of the model is given by

$$\theta = (\omega_1, \omega_2, \alpha_1, \alpha_2, \rho, \gamma)' \tag{2.5}$$

with $\omega_i > 0$, $\alpha_i \geq 0$, $|\rho| < 1$ and $\gamma \geq 0$. And the true value of θ is as usual denoted θ_0.

We make the following assumption for the exogenous series:

Assumption eXo: Assume the 3*-dimensional* $(z_t', x_t)'$ *series is i.i.d.* $\mathrm{N}(0, \Upsilon)$ *with*

$$\Upsilon = \begin{pmatrix} I_2 & \sigma_{zx} \\ \sigma_{xz} & \sigma_x^2 \end{pmatrix}, \tag{2.6}$$

where $\sigma_{zx} = \mathrm{cov}(z_t, x_t)$ *is* (2×1) *dimensional.*

Under Assumption eXo it holds that $v_t = (y_{1,t}, y_{2,t}, x_t)'$ is a three-dimensional Markov-chain with conditional density of v_t given v_{t-1},

$$f(v_t|v_{t-1}) = \frac{1}{\left(\sqrt{2\pi}\right)^3} \det(\Sigma_t)^{-1/2} \exp\left\{-\frac{1}{2} v_t' \Sigma_t^{-1} v_t\right\}, \text{ where} \tag{2.7}$$

$$\Sigma_t = \begin{pmatrix} \Omega_t & \sigma_{zx} \\ \sigma_{xz} & \sigma_x^2 \end{pmatrix}. \tag{2.8}$$

It follows that the drift-criterion from Markov-chain theory can be applied with drift function,

$$\delta(v) = 1 + ||v||^2 = 1 + v'v, \tag{2.9}$$

where $v = (y_1, y_2, x)'$. From the same Markov-chain theory, it follows that v_t is stationary and ergodic with $E||v_t||^2 < \infty$, if it holds that

$$E(\delta(v_t) \mid v_{t-1} = v) < \beta\delta(v), \tag{2.10}$$

for some $0 < \beta < 1$ and $||v||^2 = v'v > M$, where M is some large number.

[8] Drift criterion:

[8.1] Set $\sigma_{zx} = 0$, and show that

$$E(\delta(v_t) \mid v_{t-1} = v) = 1 + \mathrm{tr}(\Omega_t) + \sigma_x^2 = c + 2\gamma x^2 + \alpha_1 y_1^2 + \alpha_2 y_2^2, \tag{2.11}$$

where $c = 1 + \sigma_x^2 + \omega_1 + \omega_2$.

[8.2] State a condition on $(\alpha_1, \alpha_2, \gamma)$ such that (2.10) holds.

[8.3] What happens if $\sigma_{zx} \neq (0, 0)'$?

[9] Testing $H_x : \gamma = 0$ by a likelihood ratio (LR) test, $LR(\gamma = 0)$:
With $\ell_T(\theta)$ the log-likelihood function for $\{Y_t, x_t\}_{t=1}^T$, with initial values x_0 and Y_0 fixed, we initially wish to show that as $T \to \infty$,

$$\frac{1}{\sqrt{T}} s_T(\theta_0) = \frac{1}{\sqrt{T}} \partial\ell_T(\theta) / \partial\gamma|_{\theta=\theta_0} \to_D N\left(0, \sigma_\gamma^2\right) \tag{2.12}$$

where $\sigma_\gamma^2 = 2E\left(x_{t-1}^4 / \left(\sigma_{1,t}^4 + \sigma_{2,t}^4\right)\right)$.

[9.1] Show that (2.12) holds under Assumption eXo with $\gamma_0 > 0$. Discuss if a milder condition of Ass. eXo could be used.

[9.2] With $\gamma_0 > 0$ discuss what this implies for inference and testing in this MGARCH-X model.

[9.3] If $\gamma_0 = 0$, under which conditions will (2.12) hold?

[9.4] What are the implications of $\gamma_0 = 0$ for inference and testing?

[10] Bootstrap:

[10.1] Propose a bootstrap algorithm for the LR-based testing of $\gamma = 0$.

[10.2] Suppose instead one wants to test the hypothesis of $\alpha_1 = \alpha_2 = 0$. Discuss how you would modify the bootstrap algorithm. And if you can, discuss what issues there may be if one wants to show that the bootstrap LR statistic, LR^*, converges weakly (in probability) to the relevant limiting distribution. Issue: e.g. $\gamma_0 = 0$ or $\gamma_0 \neq 0$.